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DOUBLE DIRICHLET AVERAGE OF M - FUNCTION AND FRACTIONAL DERIVATIVE

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ABSTRACT

The object of the preset paper is to establish a result of Double Dirichlet average of M - function [14] by using fractional derivative.

KEYWORDS AND PHRASES: Dirichlet average, M - function and Fractional calculus operators.
Mathematics Subject Classification: 26A33, 33A30, 33A25 and 83C99.

INTRODUCTION

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

Recently, Gupta and Agarwal [9, 10] found that averaging process is not altogether new but directly connected with the old theory of fractional derivative. Carlson overlooked this connection whereas he has applied fractional derivative in so many cases during his entire work. Deora and Banerji [6] have found the double Dirichlet average of e^x by using fractional derivatives and they have also found the Triple Dirichlet Average of x^t by using fractional derivatives [7].

In the present paper the Dirichlet average of **M – function [14]** has been obtained.

DEFINITIONS

Some definitions which are necessary in the preparation of this paper.

Standard Simplex in $R^n, n \geq 1$:

Denote the standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_{1,}u_2, u_n) \ : \ u_1 \geq 0, \dots u_n \geq 0, \ u_1 + u_2 + \cdots + u_n \leq 1\}$$

Dirichlet measure:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by $E[1]$.

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots u_{k-1}^{b_{k-1}-1}(1 - u_1 - \cdots - u_{k-1}) b_k^{-1} du_1 \dots du_{k-1}$$

Will be called a Dirichlet measure.

Here

$$B(b) = B(b1, \dots bk) = \frac{\Gamma(b_1) \dots \Gamma(b_k)}{\Gamma(b_1 + \cdots + b_k)},$$

$$C_> = \{z \in z : z \neq 0, |ph\, z| < {}^{\pi}/_{2}\},$$

Open right half plane and $C_>$k is the k^{th} Cartesian power of $C_>$

Dirichlet Average[1, p.75]:

Let Ω be the convex set in $C_{>}$, let $z = (z_1, \dots, z_k) \in \Omega^{\mathrm{k}}, \mathrm{k} \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots, z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b,z) = \int_E f(u.z)d\,\mu_b(u) \tag{2.3}$$

F is the Dirichlet measure of f with variables
$z = (z_1, \dots, z_k)$ and parameters $b = (b_1, \dots b_k)$.
Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \cdots - u_{k-1.}$$

If $k = 1$, define $F(b,z) = f(z)$.

$\mathbf{M}-$ function[14] :
We give the new special function, called $\mathbf{M}-$ function, which is the most generalization of New Generalized Mittag-Leffler Function . Here, we give the notation and the definition of the New Special $\mathbf{M}-$ function, introduced by the author [14] as follows:

$${}_{p}^{\alpha,\beta,\gamma,\delta,\rho}\mathbf{M}_{q}^{k_1,\dots k_p,l_1,\dots l_q;c}(t) = \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots\ (a_p)_n\,(\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{(ct)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} \tag{2.4}$$

There are p upper parameters $a_{1,} a_{2,} \dots a_p$ and q lower parameters $b_{1,}\ b_{2,} \dots b_q, \alpha,\ \beta,\ \gamma, \delta, \rho \epsilon C, Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > 0$, $Re(\delta) > 0$, $Re(\rho) > 0$, $Re(\alpha\gamma - \beta) > 0$and $(a_j)_k\ (b_j)_k$ are pochammer symbols and $k_1, \dots k_p$, $l_{1,} \dots l_q$are constants. The function (1) is defined when none of the denominator parameters $b_j s,\ j = 1,2, \dots q$ is a negative integer or zero. If any parameter a_j is negative then the function (1) terminates into a polynomial in (t).

Fractional Derivative [8, p.181]:
The theory of fractional derivative with respect to an arbitrary function has been used by Erdelyi[8]. The most common definition for the fractional derivative of order α found in the literature on the “Riemann-Liouville integral” is

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \tag{2.5}$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$.

Average of M-Function (from [4]):
let μ^b be a Dirichlet measure on the standard simplex E in R^{k-1}; k$\geq$ 2. For every $z \in C^k$

$$S(b,z) = \int_E {}_{p}^{\alpha,\beta,\gamma,\delta,\rho}\mathbf{M}_{q}^{k_1,\dots k_p,l_1,\dots l_q;c}\ (u,z;v)d\mu_b\ (u) \tag{2.6}$$

If $k = 1, S = (b,z) = M\ (u,z;v)$.

Double averages of functions of one variable (from [1, 2]): let z be a $k \times x$ matrix with complex elements z_{ij}. Let $u = (u_1, \dots u_k) and\ v = (v_1, \dots v_k)$ be an ordered k-tuple and x-tuple of real non-negative weights $\sum u_i = 1\ and\ \sum v_j = 1$, respectively.
Define

$$u.z.v = \sum_{i=1}^{k} \sum_{j=1}^{x} u_i z_{ij} v_j \tag{2.7}$$

If z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $(z_{11}, \dots z_{kx})$, denote by $H(z)$.

Let $b = (b_1, \dots b_k) be$ an ordered k –tuple of complex numbers with positive real part (Re(b) > 0) and similarly for β=$(\beta_{1,} \dots \beta_x)$. Then we define $d\mu_b(u) and\ d\mu_b(v)$.

Let f be the holomorphic on a domain D in the complex plane, If $Re(b) > 0, Re(\beta) > 0 and H(z) \subset D$, we define

$$F(b,z,\beta) = \iint f(u,z,v)\, d\mu_b(u) d\mu_b(v) \tag{2.8}$$

Corresponding to the particular function $coshx$, $z^t and\ e^z$, we define,

$$S(b,z,\beta) = \iint M\ (u,z,v;\alpha) d\mu_b(u)\, d\mu_b(v) \tag{2.9}$$

$$R_t(b,z,\beta) = \iint (u,z,v)^t d\mu_b(u)\, d\mu_b(v) \tag{2.10}$$

$$S(b,z,\beta) = \iint (e)^{u.z.v} d\mu_b(u)\, d\mu_b(v) \tag{2.11}$$

Main Results and Proof

Theorem: Following equivalence relation for Double Dirichlet Average is established for $(k = x = 2)$ of **M** - Function

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'} D_{x-y}^{-\rho'}\, {}^{\alpha,\beta,\gamma,\delta,\rho}_{\quad p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}(x)\,(x-y)^{\rho-1} \tag{3.1}$$

Proof:

Let us consider the double average for ($k = x = 2$ of ${}^{\alpha,\beta,\gamma,\delta,\rho}_{\quad p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}(u.z.v)$

$$S(\mu,\mu';z;\rho,\rho') = \int_0^1\int_0^1 {}^{\alpha,\beta,\gamma,\delta,\rho}_{\quad p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}\ (u,z,v)\ dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v)$$

$$= \sum_{n=o}^{\infty} \frac{(a_1)_n\ \dots\ (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} \int_0^1\int_0^1 [u.z.v]^{(n+\gamma)\alpha-\beta-1} dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v) \quad (3.2)$$

$Re(\mu) = 0, Re(\mu') = 0, Re(\rho) > 0, Re(\rho') > 0$ and

$$u.z.v = \sum_{i=1}^{2}\sum_{i=1}^{2}(u_i z_{ij} v_j) = \sum_{i=1}^{2}[u_i(z_{i1}v_1 + z_{i2}v_2)]$$
$$= [u_1 z_{11} v_1 + u_1 z_{12} v_2 + u_2 z_{21} v_1 + u_2 z_{22} v_2]$$

let $z_{11} = a, z_{12} = b, z_{21} = c, z_{22} = d$ and $\begin{cases} u_1 = u, & u_2 = 1-u \\ v_1 = v, & v_2 = 1-v \end{cases}$

thus $z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$v.z.v = uva + ub(1-v) + (1-u)cv + (1-u)d(1-v)$$
$$= uv(a-b-c+d) + u(b-d) + v(c-d) + d$$

$$dm_{\mu,\mu'}(u) = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'} u^{\mu-1}(1-u)^{\mu'-1}du$$

$$dm_{\rho,\rho'}(v) = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} v^{\rho-1}(1-v)^{\rho'-1}dv$$

Putting these values in (3.2), we have,

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \times \sum_{n=o}^{\infty} \frac{(a_1)_n\ \dots\ (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)}$$
$$\times \int_0^1\int_0^1 [uv(a-b-c+d) + u(b-d) + v(c-d) + d]^{n(n+\gamma)\alpha-\beta-1n}\, u^{\mu-1}(1-u)^{\mu'-1} v^{\rho-1}(1$$
$$- v)^{\rho'-1} dudv$$

In order to obtained the fractional derivative equivalent to the above integral, we assume $a = c = x; b = d = y$ then

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{n=o}^{\infty}\frac{(a_1)_n\ \dots\ (a_p)_n\,(\gamma)_n}{(b_1)_n\dots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{k_1^n\dots k_p^n}{l_1^n\dots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)}$$

$$\times\int_0^1\int_0^1[uv(x-y)+y]^{n(n+\gamma)\alpha-\beta-1}\,u^{\mu-1}(1-u)^{\mu'-1}v^{\rho-1}(1-v)^{\rho'-1}dudv$$

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{n=o}^{\infty}\frac{(a_1)_n\ \dots\ (a_p)_n\,(\gamma)_n}{(b_1)_n\dots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{k_1^n\dots k_p^n}{l_1^n\dots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)}$$

$$\times\int_0^1[uv(x-y)+y]^{n(n+\gamma)\alpha-\beta-1}\,v^{\rho-1}(1-v)^{\rho'-1}dv$$

Putting $v(x-y)=t$, we obtain

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{n=o}^{\infty}\frac{(a_1)_n\ \dots\ (a_p)_n\,(\gamma)_n}{(b_1)_n\dots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{k_1^n\dots k_p^n}{l_1^n\dots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)}$$

$$\times\int_0^{x-y}[y+t]^{(n+\gamma)\alpha-\beta-1}\left(\frac{t}{x-y}\right)^{\rho-1}\left(1-\frac{t}{x-y}\right)^{\rho'-1}\frac{dt}{(x-y)}$$

$$= \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}(x-y)^{1-\rho-\rho'}\sum_{n=o}^{\infty}\frac{(a_1)_n\ \dots\ (a_p)_n\,(\gamma)_n}{(b_1)_n\dots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{k_1^n\dots k_p^n}{l_1^n\dots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)}\int_0^{x-y}[y$$

$$+t]^{(n+\gamma)\alpha-\beta-1}\,(t)^{\rho-1}(x-y-t)^{\rho'-1}dt$$

On changing the order of integration and summation, we have

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}(x-y)^{1-\rho-\rho'}\int_0^{x-y}{}^{\alpha,\beta,\gamma,\delta,\rho}{}_{p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}(y+t)\,(t)^{\rho-1}(x-y-t)^{\rho'-1}dt$$

Using definition of fractional derivative (2.4), we get

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'}D_{x-y}^{-\rho'}\,{}^{\alpha,\beta,\gamma,\delta,\rho}{}_{p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}(x)\,(x-y)^{\rho-1}$$

This is complete proof of (3.1).

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